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REGULARITY OF ISOMETRIC IMMERSIONS OF POSITIVELY CURVED RIEMANNIAN MANIFOLDS AND ITS ANALOGY WITH CR GEOMETRY

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Abstract

Let M be an n-dimensional Riemannian manifold and F be an isometric immersion of M into $\mathbf{R}^{n+1}$. It is shown that under certain conditions on the sign of principal curvatures of $F(M)$, F satisfies an over-determined system of elliptic partial differential equations after one adds the scalar curvature equation. As a corollary, if M is an analytic manifold of positive sectional curvature, F is analytic and uniquely determined by $F(P)$ and $dF(P)$ at a reference point P of M. An analogous problem in CR geometry is proposed.

0. Introduction and statement of the main results

We are concerned in this paper with the regularity and the uniqueness of isometric immersions of n-dimensional Riemannian manifolds into $\mathbf{R}^{n+1}$. We deal with analytic (C^ω) manifolds. However, one can get a C^∞ version of this paper by replacing every C^ω by C^∞. Consider first the following well-known fact: If M is a C^ω connected Riemannian manifold and F is a continuously differentiable isometry of M onto another C^ω Riemannian manifold $\tilde{M}$, then F is C^ω. Moreover, if O is a point of M, then F is uniquely determined by $F(O)$ and the first partial derivatives of F at O. The reason is that locally F can be expressed as a linear mapping between the normal coordinates of M and $\tilde{M}$ near O and $F(O)$, respectively. Analyticity and uniqueness with respect to the initial data at one point follow from the viewpoint of the local equivalence problem also under the assumption $F \in C^2$ (cf. [2] and [4]). Our question is whether one can remove the hypothesis of analyticity of $\tilde{M}$ when $\tilde{M}$ is a hypersurface in a Euclidean space; namely,

Question 1. Let M be an n-dimensional C^ω Riemannian manifold and $F = (f^1, \cdots, f^{n+1})$ be a C^k, $k \gg 0$, isometric immersion of M into $\mathbf{R}^{n+1}$. Then will F be C^ω? And will F be uniquely determined by $F(O)$ and the first partial derivatives of F at a point?

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The following example shows that if M is flat, F is neither C^ω nor determined by its partial derivatives at a point.

Example 1. Let $\gamma(s) = (y^1(s), y^2(s))$ be a plane curve parametrized by arclength s. If γ is C^∞ but not C^ω, the mapping $(s,t) \to (y^1(s), y^2(s), t)$ is a C^∞ isometric immersion of $\mathbf{R}^2$ into $\mathbf{R}^3$, which is not C^ω. Thus we see that certain curvature conditions must be imposed. We here prove

Theorem 1. *Let M be a C^ω Riemannian manifold of dimension $n \geq 2$ and let $F = (f^1, \cdots, f^{n+1})$ be a C^2 isometric immersion of M into $\mathbf{R}^{n+1}$. Let $O \in M$, $\tilde{O} = F(O)$ and $\tilde{M} = F(M)$. Let $\lambda_1, \cdots, \lambda_n$ be the principal curvatures of $\tilde{M}$ at $\tilde{O}$ and let*

$$\Lambda_k = \sum_{j \neq k} \lambda_j \quad \text{for each } k = 1, \cdots, n.$$

Suppose that each λ_j, $j = 1, \cdots, n$, is nonzero and $\Lambda_1, \cdots, \Lambda_n$ are all positive or all negative. Then F is C^ω on a neighborhood of O.

The idea of the proof is to show that $(f^1, \cdots, f^{n+1})$ satisfies a system of nonlinear partial differential equations of second order, where each equation is C^ω in its arguments and the system is elliptic at $(f^1, \cdots, f^{n+1})$. Then the analyticity of F follows from the theory of elliptic partial differential equations (cf. [7, p. 15]). A detailed proof will be presented in §1. In the statement of Theorem 1, $\mathbf{R}^{n+1}$ can be replaced by a C^ω Riemannian manifold of dimension $n+1$, which can be proved by a slight modification of our proof of Theorem 1.

By combining Theorem 1 and classical rigidity theorems for hypersurfaces in $\mathbf{R}^{n+1}$ we can prove the following theorems on the regularity and uniqueness of isometric immersions.

Theorem 2. *Suppose that M is a C^ω connected Riemannian manifold of dimension $n \geq 3$ of positive sectional curvature and $F\colon M \to \mathbf{R}^{n+1}$ is a C^2 isometric immersion. Then F is C^ω. Moreover, if F' is another such isometric immersion there exists an isometry τ of $\mathbf{R}^{n+1}$ such that $F' = \tau \circ F$.*

Theorem 3. *Suppose that M is a 2-dimensional compact C^ω Riemannian manifold of positive Gaussian curvature and $F\colon M \to \mathbf{R}^3$ is a C^2 isometric immersion. Then F is C^ω. Moreover, if F' is another such isometric immersion there exists an isometry τ of $\mathbf{R}^3$ such that $F' = \tau \circ F$.*

1. Proof of the theorems

Proof of Theorem 1. Showing analyticity of a mapping is a local problem, so let M be a "germ" of a C^ω manifold at a reference point $O \in M$. Let $(y^1, \cdots, y^{n+1})$ be the standard coordinates of $\mathbf{R}^{n+1}$ and write $F =$

$(f^1, \cdots, f^{n+1})$ coordinatewise. We may assume that $\tilde{O}$ is the origin of $\mathbf{R}^{n+1}$ and $\tilde{M}$ is tangent to the plane $y^{n+1} = 0$. Let N be a unit normal vector field of $\tilde{M}$ and $\tilde{A}$ be the second fundamental form; namely,

$$\tilde{A}(X, Y) \equiv \langle \nabla'_X N, Y \rangle \ \forall \text{ tangent vectors } X, Y \text{ of } \tilde{M} \text{ at } \tilde{O},$$

where ∇' is the covariant differentiation of $\mathbf{R}^{n+1}$. The eigenvalues $\lambda_1, \cdots, \lambda_n$ of the linear transformation $v \to \nabla'_v N$ are called the principal curvatures at $\tilde{O}$. Let $v_1, \cdots, v_n$ be the orthonormal eigenvectors which correspond to the principal curvatures $\lambda_1, \cdots, \lambda_n$. Let $\{e_1, \cdots, e_n\}$ be an orthonormal frame over M such that $F_* e_j = v_j$ at $\tilde{O}$. We see that

$$\tilde{e}_j \equiv F_* e_j = \sum_{\nu=1}^{n+1} (e_j f^\nu) \circ F^{-1} \partial / \partial y_\nu.$$

We may assume further that

$$\tilde{e}_j = \partial / \partial y_j \quad \text{at } \tilde{O}, \ j = 1, \cdots, n.$$

Then we have

$$e_j f^\nu(O) = \begin{cases} 0 & \text{if } j \neq \nu, \\ 1 & \text{if } j = \nu. \end{cases} \tag{1.1}$$

Now let $(\tilde{\eta}_1, \cdots, \tilde{\eta}_{n+1})$ be the components of N and let $\eta_j = \tilde{\eta}_j \circ F$. To express η_j in terms of partial derivatives of $(f^1, \cdots, f^{n+1})$ consider the matrix

$$P \equiv \begin{bmatrix} e_1 f^1 & \ldots & e_1 f^{n+1} \\ \vdots & & \vdots \\ e_n f^1 & \ldots & e_n f^{n+1} \\ \eta_1 & & \eta_{n+1} \end{bmatrix} \in \mathrm{O}(n).$$

We may assume that $\eta_{n+1}(0) = 1$ so that $\det M = 1$. Choose a local coordinate system $(x_1, \cdots, x_n)$ of M such that $e_j = \partial / \partial x_j$ at 0, $j = 1, \cdots, n$. Since $M^{-1} = M^t$, each η_j is equal to its cofactor in P. Thus we have

$$\eta_j = (e_j f^{n+1}) B_j + \sum_{\lambda \neq j} e_\lambda f^{n+1} \varsigma_{j\lambda}, \qquad j = 1, \cdots, n, \tag{1.2}$$

and $\eta_{n+1} = (e_1 f^1) \cdots (e_n f^n) + \varsigma$, where B_j, $\varsigma_{j\lambda}$, ς are C^ω functions in $(x, D^\alpha f^i \colon i \neq n+1, |\alpha| \leq 1)$ such that $B_j = 1$, $\varsigma_{j\lambda} = 0$ and $\varsigma = 0$ at $(0, D^\alpha f^i(0))$.

Now let $A(x) = [A_{ij}(x)]$ be the symmetric matrix defined by

$$A_{ij}(x) = \tilde{A}(\tilde{e}_i, \tilde{e}_j) \circ F = \langle \nabla'_{\tilde{e}_i} N, \tilde{e}_j \rangle \circ F.$$

We express $A_{ij}(x)$ in terms of $(f^1, \cdots, f^{n+1})$ and their partial derivatives:

$$\nabla'_{\tilde{e}_i} N = (\tilde{e}_i \tilde{\eta}_1, \cdots, \tilde{e}_i \tilde{\eta}_{n+1}) = (e_i \eta_1, \cdots, e_i \eta_{n+1}) \circ F^{-1}.$$

But by (1.1) and (1.2) we have

$$e_i \eta_k = (e_i e_k f^{n+1}) B_k + \sum_{\lambda \neq k} (e_i e_\lambda f^{n+1}) \varsigma_{k\lambda} + C_{ik}, \ k = 1, \cdots, n,$$

and $e_i \eta_{n+1} = C_{i,n+1}$, where each C_{ik} and $C_{i,n+1}$ are C^ω functions of $(x, D^\alpha f^i$: $i \neq n+1$, $|\alpha| \leq 2)$, and thus we see that

$$A_{ij}(x) = (e_i e_j f^{n+1}) B_j (e_j f^j) + \sum (e_\lambda e_\mu f^\nu) \varsigma^\nu_{\lambda\mu}, \tag{1.3}$$

where each $\varsigma^\nu_{\lambda\mu}$ is a C^ω function in $(x, D^\alpha f^i \colon |\alpha| \leq 1)$, which vanishes at $(0, D^\alpha f^i(0))$. Since $\tilde{e}_j = v_j$ at $\tilde{O}$, $j = 1, \cdots, n$, which is the eigenvector of the linear transformation $v \to \nabla'_v N$, we have

$$A_{ij}(0) = \begin{cases} 0 & \text{if } i \neq j, \\ \lambda_j & \text{if } i = j. \end{cases} \tag{1.4}$$

Now let S and $\tilde{S}$ be the scalar curvatures of M and $\tilde{M}$, respectively. Since F is an isometry, $S(x) = \tilde{S}(F(x))$. Let $\det(A(x) - \lambda I) = \sum_{k=0}^n a_k(x) \lambda^k$ be the characteristic polynomial of A. Then $\tilde{S}(F(x)) = 2a_2(x)$ (cf. [6]). But

$$\frac{1}{2} S(x) = \frac{1}{2} \tilde{S}(F(x)) = a_2(x) = \sum_{i<j} A_{ii} A_{jj} + \sum A_{qm} A_{q'm'},$$

where each term in the second sum involves a nondiagonal entry, therefore vanishes at O by (1.4). Substituting (1.3) for the A_{ij}'s we have

$$\begin{aligned} \frac{1}{2} S(x) = & \sum_{i<j} (e_i e_i f^{n+1})(e_j e_j f^{n+1}) B_i B_j (e_i f^i)(e_j f^j) \\ & + \sum (e_\lambda e_\mu f^\nu)(e_{\lambda'} e_{\mu'} f^{\nu'}) \varsigma^{\nu\nu'}_{\lambda\mu\lambda'\mu'}, \end{aligned} \tag{1.5}$$

where each $\varsigma^{\nu\nu'}_{\lambda\mu\lambda'\mu'}$ is a C^ω function in $(x, D^\alpha f^i \colon |\alpha| \leq 1)$, which vanishes at $(0, D^\alpha f^i(0))$. (1.5) is an equation for $(f^1, \cdots, f^{n+1})$. To get other equations, we observe that the first n rows of P are orthonormal and therefore $(e_i f^1)(e_j f^1) + \cdots + (e_i f^{n+1})(e_j f^{n+1}) = \delta_{ij}$ (Kronecker's delta).

Apply e_i to the above to get

$$\begin{aligned} & (e_i e_i f^1)(e_j f^1) + (e_i f^1)(e_i e_j f^1) + \cdots + (e_i e_i f^{n+1})(e_j f^{n+1}) \\ & + (e_i f^{n+1})(e_i e_j f^{n+1}) = 0. \end{aligned} \tag{1.6}$$

We shall show that the system of equations (1.6) with $i, j = 1, \cdots, n$ and (1.5) is elliptic at $(f^1, \cdots, f^{n+1})$. Express (1.6) and (1.5) in terms of coordinates $(x_1, \cdots, x_n)$.

$$
\begin{aligned}
&\left(\frac{\partial}{\partial x_i}\right)^2 f^1 \frac{\partial f^1}{\partial x_j} + \frac{\partial f^1}{\partial x_i}\left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} f^1\right) \\
&\qquad + \cdots + \left(\frac{\partial}{\partial x_i}\right)^2 f^{n+1} \frac{\partial f^{n+1}}{\partial x_j} + \frac{\partial f^{n+1}}{\partial x_i}\left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} f^{n+1}\right) \\
&\qquad + \sum_{\lambda\mu\nu}\left(\frac{\partial}{\partial x_\lambda}\frac{\partial}{\partial x_\mu} f^\nu\right)\varsigma^\nu_{\lambda\mu} \equiv H_{ij}(x, D^\alpha f^k) = 0,
\end{aligned} \tag{1.6'}
$$

$$
\begin{aligned}
&\sum_{i<j}\left(\frac{\partial}{\partial x_i}\right)^2 f^{n+1}\left(\frac{\partial}{\partial x_j}\right)^2 f^{n+1} B_i B_j \frac{\partial f^i}{\partial x_i}\frac{\partial f^j}{\partial x_j} \\
&\qquad + \sum\left(\frac{\partial}{\partial x_\lambda}\frac{\partial}{\partial x_\mu} f^\nu\right)\left(\frac{\partial}{\partial x'_\lambda}\frac{\partial}{\partial x'_\mu} f^{\nu'}\right)\varsigma^{\nu\nu'}_{\lambda\mu\lambda'\mu'} - \frac{1}{2}S(x) \\
&\equiv H(x, D^\alpha f^k) = 0,
\end{aligned} \tag{1.5'}
$$

where each $\varsigma^\nu_{\lambda\mu}, \varsigma^{\nu\nu'}_{\lambda\mu\lambda'\mu'}$ is a C^ω function of $(x, D^\alpha f^k \colon |\alpha| \le 1)$ and vanishes at $(0, D^\alpha f^k(0))$. These ς's are different from the ς's that previously appeared. Consider the linear partial differential operators L_{ij} and L defined by

$$
L_{ij} w = \sum_{\substack{|\alpha|\le 2 \\ k=1,\cdots,n+1}} \frac{\partial H_{ij}}{\partial(D^\alpha f^k)} D^\alpha w^k, \quad L w = \sum_{\substack{|\alpha|\le 2 \\ k=1,\cdots,n+1}} \frac{\partial H}{\partial(D^\alpha f^k)} D^\alpha w^k,
$$

where $w = (w^1, \cdots, w^{n+1})$. Then L_{ij} and Lw are of the following form:

$$
\begin{aligned}
L_{ij} w = {} & E_{ij}\left(\frac{\partial}{\partial x_i}\right)^2 w^j + G_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} w^i \\
& + \sum \varsigma^\nu_{\lambda\mu}\frac{\partial}{\partial x_\lambda}\frac{\partial}{\partial x_\mu} w^\nu + \text{ lower order terms},
\end{aligned} \tag{1.7}
$$

$$
\begin{aligned}
L w = {} & \frac{1}{2}\sum_{j\ne i}\left(\frac{\partial}{\partial x_j}\right)^2 f^{n+1} K_{ij}\left(\frac{\partial}{\partial x_i}\right)^2 w^{n+1} \\
& + \sum \tilde{\varsigma}^\nu_{\lambda\mu}\frac{\partial}{\partial x_\lambda}\frac{\partial}{\partial x_\mu} w^\nu + \text{ lower order terms},
\end{aligned} \tag{1.8}
$$

where E_{ij}, G_{ij}, K_{ij} are C^ω functions in $(x, D^\alpha f^k \colon |\alpha| \le 1)$ with values 1 at $(0, D^\alpha f^k(0))$, each $\varsigma^\nu_{\lambda\mu}$ is a C^ω function of $(x, D^\alpha f^k \colon |\alpha| \le 1)$ which vanishes at $(0, D^\alpha f^k(0))$ and each $\tilde{\varsigma}^\nu_{\lambda\mu}$ is a C^ω function of $(x, D^\alpha f^k\colon |\alpha| \leq 2)$ which vanishes at $(0, D^\alpha f^k(0))$. These ς's are different from those which appeared previously. Consider the principal symbol $\sigma(x,\xi)$ of the system (1.7), (1.8) (cf. [7]). $\sigma(x,\xi)$ is a matrix of size $(n^2+1)\times(n+1)$. We decompose $\sigma(x,\xi)$ into $n+1$ blocks as

$$\sigma(x,\xi) = \begin{bmatrix} \sigma_1(x,\xi) \\ \vdots \\ \sigma_n(x,\xi) \\ \sigma_{n+1}(x,\xi) \end{bmatrix},$$

where $\sigma_j(x,\xi)$, $j = 1, \cdots, n$, is the principal symbol matrix of the system (1.7) with $i = 1, \cdots, n$ and fixed j, and σ_{n+1} is that of (1.8). Then for $j = 1, \cdots, n$,

$$\sigma_j(0,\xi) = \begin{bmatrix} \xi_1\xi_j & 0 & \cdots & 0 & \xi_1^2 & 0 & \cdots & \cdots & 0 \\ 0 & \xi_2\xi_j & \cdots & 0 & \xi_2^2 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \xi_n^2 & 0 & \cdots & \xi_n\xi_j & 0 \\ & & & & j\text{th column} & & & & \end{bmatrix}_{n\times(n+1)}$$

Thus we see that $\forall\xi \neq 0$ the first n columns of $\sigma(0,\xi)$ are linearly independent. But the last entry of $\sigma_{n+1}(0,\xi)$ is

$$\frac{1}{2}\sum_{j=1}^{n}\left(\sum_{i\neq j}\frac{\partial^2 f^{n+1}}{\partial x_i^2}(0)\right)\xi_j^2$$

$$= \frac{1}{2}\sum_{j=1}^{n}\left(\sum_{i\neq j} e_i e_i f^{n+1}(0)\right)\xi_j^2,$$

where $e_i e_i f^{n+1}(0) = A_{ii}(0) = \lambda_i$, by (1.3) and (1.4),

$$= \frac{1}{2}\sum_{j=1}^{n}\left(\sum_{i\neq j}\lambda_i\right)\xi_j^2 = \frac{1}{2}\sum_{j=1}^{n}\Lambda_j\xi_j^2,$$

which is nonzero $\forall\xi\neq 0$ by the hypothesis of the theorem. Therefore, $(n+1)$ columns of $\sigma(0,\xi)$ are linearly independent.

Now regard $\sigma(x,\xi)$ as a matrix valued function on $\Omega\times S^{n-1}$, where Ω is a neighborhood of the origin of $\mathbf{R}^n$. Since S^{n-1} is compact we see that there is a neighborhood $\Omega' \subset \Omega$ of the origin of $\mathbf{R}^n$ so that $\sigma(x,\xi)$ has rank $n+1$, $\forall x\in\Omega'$, $\forall\xi\in S^{n-1}$. This completes the proof of Theorem 1.

Let v_j, $j = 1, \cdots, n$, be as in the proof of Theorem 1. Then the sectional curvature $K(v_i\wedge v_j)$ of the plane $v_i\wedge v_j$ is given by $K(v_i\wedge v_j) = \lambda_i\lambda_j$ (cf. [6]). Therefore, if M (and hence $\tilde{M}$) has positive sectional curvature all the principal curvatures $\lambda_1, \cdots, \lambda_n$ are of the same sign. Thus, analyticity of F in Theorems 2 and 3 follows from Theorem 1. The uniqueness part of Theorems 2 and 3 follows from the following rigidity theorems. Recall that a hypersurface M_1 is said to be rigid if for any isometry τ_0 of M_1 onto another hypersurface M_2 there exists an isometry τ of $\mathbf{R}^{n+1}$ such that $\tau_0 = \tau$ on M_1.

Theorem [5, p. 120]. *If $n \geq 3$ and M is an oriented hypersurface in $\mathbf{R}^{n+1}$ with positive sectional curvature, then M is rigid.*

Theorem (*Cohn-Vossen* [5, p. 122]). *A compact surface of positive Gaussian curvature is rigid.*

2. Analogy with CR geometry

The author has been motivated from the following analogous problem in CR geometry. We refer to [3] for definitions.

Question 2. Let M be a C^ω CR manifold of dimension $2n+d$ of CR codimension d and $F\colon M \to \mathbf{C}^{n+d}$ is a CR immersion of differentiability C^k, $k \gg 0$. Then will F be C^ω?

The following example shows that certain "curvature" conditions must be imposed on M.

Example 2. Let $M = \mathbf{C}^1 \times \mathbf{R}^1 = \{(x+iy, t)\}$ and let $\gamma(t) = u(t)+iv(t)$ be a C^∞, but not C^ω, complex valued function. Then the mapping $(x+iy, t) \to (x+iy, \gamma(t)) \in \mathbf{C}^2$ is a C^∞ CR immersion which is not C^ω. Observe that M is Levi flat.

Let us now consider the cases where M is a C^ω hypersurface in $\mathbf{C}^{n+d}$. Let $F = (f^1, \cdots, f^{n+d})$ be a system of CR functions of M where dF is of the maximal rank at each point of M. We shall call such F a local CR diffeomorphism instead of CR immersion. Then the following are equivalent:

(i) Every C^k local CR diffeomorphism F is C^ω.

(ii) For a C^k local CR diffeomorphism F and $P \in M$, there exist a neighborhood Ω_F of P in $\mathbf{C}^{n+d}$ so that F extends to a biholomorphic mapping of Ω_F.

(iii) For a C^k CR function f and $P \in M$, there exists a neighborhood Ω_f of P in $\mathbf{C}^{n+d}$ so that f extends to a holomorphic function of Ω_f.

See [1] for related results.

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